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Extension of Wheeler–Feynman quantum theory to the relativistic domain I. Scattering processes

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Abstract. The quantization of Wheeler–Feynman direct interparticle action electrodynamics is refined and extended to the relativistic domain, using the S matrix technique developed previously. The analysis is effected on the exact expression for S, rather than the perturbation expansion. Source particles are treated as indistinguishable fermions with antiparticles. The response of the universe is incorporated in a general way, without allusion to specific absorber mechanisms. All the results of conventional QED emerge, provided one makes certain cosmological assumptions.

Introduction

Recently, quantum mechanical descriptions of the direct interparticle action electrodynamics of Wheeler and Feynman (1945, 1949) have been given (Hoyle and Narlikar 1969, 1971, Davies 1970, to be referred to as A). Although fundamentally symmetric in the presence of both advanced and retarded electromagnetic fields, which are here devoid of independent mechanical status, these analyses have succeeded in demonstrating that all the results of conventional QED emerge also from the Wheeler–Feynman theory, provided certain cosmological conditions are realized.

The classical Wheeler–Feynman theory derives from the Fokker action

$$J = -\sum_i \int_{\sigma_1}^{\sigma_2} m_i \left(g_{\mu\nu} \frac{\mathrm{d}z_i^\mu}{\mathrm{d}\lambda_i} \frac{\mathrm{d}z_i^\nu}{\mathrm{d}\lambda_i} \right)^{1/2} \mathrm{d}\lambda_i$$

$$-\tfrac{1}{2} \sum_{i \neq j} \sum e_i e_j \int_{(\sigma_1)}^{(\sigma_2)} \int g_{\mu\nu} \frac{\mathrm{d}z_i^\mu}{\mathrm{d}\lambda_i} \bar{D}(z_i - z_j) \frac{\mathrm{d}z_j^\nu}{\mathrm{d}\lambda_j} \mathrm{d}\lambda_i \, \mathrm{d}\lambda_j. \tag{1}$$

Equation (1) is the total action for an assembly of electromagnetically interacting particles of charge and mass e and m, respectively. z labels the world lines and the λ are monotonically increasing parameters along the respective z. The metric tensor is $g_{\mu\nu}$ and the σ are space-like surfaces outside of which δJ vanishes. The first term is just the total free particle action and the second represents all two particle interactions counted just once for each pair of particles. There is no term in the action for the free electromagnetic field as it is not a separate mechanical system in this theory. We distinguish an interaction Lagrangian for the particle i

$$L_i = e_i \sum_{j \neq i} e_j \int g_{\mu\nu} \frac{\mathrm{d}z_i^\mu}{\mathrm{d}\lambda_i} \bar{D}(z_i - z_j) \frac{\mathrm{d}z_j^\nu}{\mathrm{d}\lambda_j} \mathrm{d}\lambda_j \tag{2}$$

and a direct particle four potential due to the source j

$$\bar{A}_\mu(x) = e_j \int g_{\mu\nu} \bar{D}(x - z_j) \frac{\mathrm{d}z_j^\nu}{\mathrm{d}\lambda_j} \mathrm{d}\lambda_j. \tag{3}$$

Canonical quantization techniques cannot be applied to a theory derived from a Lagrangian such as (2), which is nonlocal in time, and the procedure adopted in A was to deal directly with the S matrix perturbation expansion†

$$S = \sum_n \frac{(-\mathrm{i})^n}{n!} \int \ldots \int P[\mathscr{L}(x_1)\ldots\mathscr{L}(x_n)]\, \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n \tag{4}$$

where P is the time ordering operator.

The S matrix is the usual starting point in conventional quantum electrodynamics where the electromagnetic field is also a quantized mechanical system. In this case one chooses

$$\mathscr{L}(x) = j^\mu(x)A_\mu(x) \tag{5}$$

where j^μ is an appropriate current four vector and A_μ is the four potential operator of the quantized electromagnetic field. The right hand side of (5) then describes the basic particle–field interaction which is the starting point of conventional QED.

Each scattering amplitude taken between an initial and final state may be associated with Feynman type graphs which correspond to different electrodynamic processes. These graphs contain internal and external photon lines which represent virtual and real photons respectively. The presence of such a line is associated with a certain factor in the S matrix, which in the conventional theory may be determined by applying the usual quantum rules to the field A_μ. In particular we need the following results

$$\langle 0|A_\mu(x)A_\nu(x')|0\rangle = -\mathrm{i}D^+(x-x')g_{\mu\nu} \tag{6}$$

$$\langle 0|P[A_\mu(x)A_\nu(x')]|0\rangle = -\mathrm{i}D_\mathrm{F}(x-x')g_{\mu\nu}. \tag{7}$$

However, if we wish to use the Wheeler–Feynman theory, A_μ is not quantized, and one must recover the above mentioned factors from the response of the universe.

In A it was shown how we could use

$$\mathscr{L}(x) = \int j_{(a)}{}^\mu(x)\bar{D}(x-x')j_{(b)\mu}(x')\, \mathrm{d}^4x\, \mathrm{d}^4x' \tag{8}$$

as our basic *first* order interaction, in place of (5), in analogy to the classical interaction Lagrangian (2), which has the form of a current–current interaction.

Having thus reconstructed the S matrix expansion, it was shown how the response of the universe enabled us to recover the basic results (6) and (7), for the transverse polarization directions. This implies that, with care over definitions, the Wheeler–Feynman theory successfully accounts for radiative transition probabilities, level shifts, etc, as described by conventional QED, to all orders of the expansion. In order to achieve these results, however, a number of simplifications were necessary.

In the present paper several improvements will be made on the existing theory. We shall work with the exact expression for S rather than the perturbation expansion (2). In A it was possible to achieve the required results without allusion to the detailed interactions of the 'laboratory' system to be described, because of a factorization into two classes; the 'laboratory' and the 'universe'. This was only possible if the direct Wheeler–Feynman interaction was omitted in the laboratory system. All interaction in this class was due to coupling through the response of the universe. This technique permitted the response of the universe to be dealt with in a simple and nonrelativistic

† We work always in the interaction representation, unless stated to the contrary.

fashion, without restricting the laboratory system to such an unsophisticated formalism, and avoided the necessity to deal with such problems as interaction between identical particles, and between particles and antiparticles, as an integral part of the Wheeler–Feynman scheme. These latter considerations could be taken over intact from the conventional theory, once the basic quantum behaviour of the A_μ field had been recovered. (This led to difficulty when dealing with free particle states.) However, the separation is clearly an idealization, as there is no criterion to determine whether a given particle (especially a free one) belongs to the 'laboratory' or the 'universe'. In this paper, then, the distinction will be removed, in the sense that the full Wheeler–Feynman coupling will be retained between all particles; identical particles and antiparticles included. The response of the universe is now incorporated in a fully relativistic and general fashion, as in derivation four of Wheeler and Feynman (1945). The procedure closely parallels that of Hoyle and Narlikar's paper II (1971), but retains the S matrix formalism. The relativistic treatment will be presented in two parts. In § 1 we shall consider how scattering processes (or, in field theory language, processes involving virtual photons only) may be described in the Wheeler–Feynman theory. In § 2 we shall consider emission processes (real photons) in which there is an irreversible energy loss to the universe.

1. Elimination of the virtual photons

The general expression for S is given by

$$S = P\exp(-\mathrm{i}J) = P\exp\{-\mathrm{i}\int \mathscr{L}(x)\,\mathrm{d}^4x\} \tag{9}$$

where the action J is now a quantum operator. Expansion of the exponential leads to equation (4). We have to decide on what to adopt for J. In A the electromagnetic interaction at each point in class one was due to interparticle coupling with a distinguishable current in class two (cf equation (5)). In order to discuss interactions between indistinguishable particles it is helpful to digress and consider an aspect of the conventional theory first discussed by Feynman (1948, 1950), that is, the elimination of the virtual photons. In the conventional theory $\mathscr{L}$ is given by (5) and S becomes

$$S = P\exp\Big(-\mathrm{i}\int j_\mu(x)A^\mu(x)\Big). \tag{10}$$

The nth order perturbation matrix element is then

$$S_n = \frac{(-\mathrm{i})^n}{n!}\int \ldots \int P[j_\mu(x_1)\ldots j_\sigma(x_n)]P[A^\mu(x_1)\ldots A^\sigma(x_n)]\,\mathrm{d}^4x_1\ldots\mathrm{d}^4x_n \tag{11}$$

where we have used the fact that the j and A commute.

Now consider only even order elements S_n where there are no photons initially and finally (virtually photons only). If we apply Wick's theorem to the second factor in the integrand of (11) we only have nonzero matrix elements in the terms in which all photon operators are contracted over. That is

$$P[\underline{A^\mu(x_1)}A^\nu(x_2)\ldots\underline{A^\rho(x_{n-1})}A^\sigma(x_n)] \tag{12}$$

together with all other similar terms in which the contractions are taken over all possible pairs of A operators. There are $n!/\{2^{n/2}\times(n/2)!\}$ of these terms and they all

give identical contributions to S_n. If we write in the photon propagator explicitly, using (7), one such term is

$$\frac{(\mathrm{i})^{n/2}}{n!}\int\ldots\int P[j_\mu(x_1)j_\nu(x_2)\ldots j_\rho(x_{n-1})j_\sigma(x_n)] \times D_{\mathrm{F}}(x_1-x_2)\ldots D_{\mathrm{F}}(x_{n-1}-x_n)g^{\mu\nu}\ldots g^{\rho\sigma}\,\mathrm{d}^4x_1\ldots\mathrm{d}^4x_n. \tag{13}$$

As all the terms are identical we may simply multiply by $n!/\{2^{n/2}\times(n/2)!\}$, and bring the c number propagators inside the P operator to give

$$\frac{(\mathrm{i})^{n/2}}{(n/2)!}2^{-n/2}\int\ldots\int P[j_\mu(x_1)D_{\mathrm{F}}(x_1-x_2)j^\mu(x_2)\ldots \times j_\rho(x_{n-1})D_{\mathrm{F}}(x_{n-1}-x_n)j^\rho(x_n)\,\mathrm{d}^4x_1\ldots\mathrm{d}^4x_n. \tag{14}$$

Now suppose instead of (10) we had started with the expression

$$P\exp\left(\tfrac{1}{2}\mathrm{i}\int\int j_\mu(x)D_{\mathrm{F}}(x-y)j^\mu(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right) \tag{15}$$

where the P operator acts on all the currents derived from the exponent. Then the $(n/2)$th order matrix element of expression (15) is given by (14). The action in the exponent of (15) has the form of a direct current–current coupling through the propagator D_{F}.

If the $j_\mu(x)$ and $j^\mu(y)$ were distinguishable currents, we should have to remove the factor $\frac{1}{2}$ from the exponent to obtain agreement with experiment. This is because we cannot permute the end points of photon lines that start and end on different currents in a Feynman graph. This gives rise to an overall factor of $2^{n/2}$ for $n/2$ photons. This situation also arises in the Hoyle and Narlikar (1969) treatment (p. 64) and may be thought of physically as follows. The interaction of two distinguishable currents, A and B, is due to the action on A in the field of B, plus the action on B in the field of A. If A and B are indistinguishable, however, the two is absent owing to the fact that a current acts on itself only once.

The most general action will consist of a collection of distinguishable and indistinguishable currents in mutual- and self-action. This is clearly described by the expression

$$P\exp\left(\mathrm{i}\sum_i\sum_j\tfrac{1}{2}\int\int j_{(i)\mu}(x)D_{\mathrm{F}}(x-y)j_{(j)}{}^\mu(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right) \tag{16}$$

where the double summation extends over all species of particle in the collection. The terms $i=j$ give the contributions such as (15), that is, self-action and interaction of indistinguishable currents. The terms $i\neq j$ account for interaction between distinguishable currents. Each such pair is counted twice in the double summation, and this will just cancel the $\frac{1}{2}$ in the exponent of (16).

Although from a physical point of view one would certainly expect that in the absence of asymptotic ($t\to\pm\infty$) excitation of the electromagnetic field the system could be regarded as a collection of interacting currents, it is imporrant to remember that the result (16) applies to identical currents also and even self-action.

We therefore have the following theorem:

Theorem 1

For all processes in QED in which the total number of real photons is zero, the conventional current–field interaction used in the S matrix for a collection of species of particles, given by

$$\sum_i \int j_{(i)}{}^{\mu}(x) A_{\mu}(x)\, \mathrm{d}^4x \tag{17}$$

may be replaced by the direct current–current interaction given by

$$\sum_i \sum_j \tfrac{1}{2} \int \int j_{(i)\mu}(x) D_{\mathrm{F}}(x-y) j_{(j)}{}^{\mu}(y)\, \mathrm{d}^4x\, \mathrm{d}^4y$$

without change in the results.

This result was originally obtained by Feynman using his own quantum mechanics in 1948. A less direct proof using the S matrix is given in Akhiezer and Beresteskii (1965, p. 302).

2. Wheeler–Feynman theory

Classical Maxwell theory supplies an expression for the total field acting on a charged particle i interacting with a collection of other charged particles j

$$\sum_{j \neq i} A^{\mathrm{ret}}_{(j)\mu} + \tfrac{1}{2}(A^{\mathrm{ret}}_{(i)\mu} - A^{\mathrm{adv}}_{(i)\mu}) \tag{18}$$

excluding a term which leads to a divergent selfenergy.

On the other hand, classical Wheeler–Feynman theory gives

$$\tfrac{1}{2} \sum_{j \neq i} (A^{\mathrm{ret}}_{(j)\mu} + A^{\mathrm{adv}}_{(j)\mu}). \tag{19}$$

The basic assertion of the absorber theory is that for a system enclosed in a light tight box, (19) may be used in place of (18) without changing the results (provided the summation includes all the box particles also). The reason is easy to see. The difference $A_{\mu}{}^{\mathrm{R}}$ between (18) and (19) is

$$A_{\mu}{}^{\mathrm{R}} = \tfrac{1}{2} \sum_{\mathrm{all}\, j} (A^{\mathrm{ret}}_{(j)\mu} - A^{\mathrm{adv}}_{(j)\mu}) = \sum_{\mathrm{all}\, j} A^{\mathrm{R}}_{(j)\mu} \qquad \text{say.}$$

But $A_{\mu}{}^{\mathrm{R}}$ clearly vanishes outside the box. Because $\Box A_{\mu}{}^{\mathrm{R}} = 0$, if it vanishes anywhere it vanishes everywhere. So (18) and (19) are the same in this case.

$A^{\mathrm{R}}_{(j)\mu}$ is sometimes called the response field of the universe for particle j. It may be written

$$A^{\mathrm{R}}_{(j)\mu}(x) = \tfrac{1}{2} \int D(x-y) J_{(j)\mu}(y)\, \mathrm{d}^4y \tag{20}$$

where $J_{(j)\mu}$ is the classical charged current of particle j. The Green function D is given by

$$D = \tfrac{1}{2}(D^{\mathrm{ret}} - D^{\mathrm{adv}}) \tag{21}$$

and is a solution of the homogeneous equation $\Box D = 0$. It may be invariantly decomposed into positive and negative frequency parts

$$D = D^{+} + D^{-} \tag{22}$$

which have the property

$$D^+(x-y) = -D^-(y-x). \tag{23}$$

The classical expression (20) makes no distinction between these components.

Returning to the quantum theory, let us now examine the integrals in the exponent of (16) in a little more detail. The Feynman propagator may be decomposed as follows

$$D_F = \bar{D} + \tfrac{1}{2}(D^+ - D^-) \tag{24}$$

where $\bar{D} = \frac{1}{2}D_{\text{ret}} + \frac{1}{2}D_{\text{adv}}$. This decomposition gives two sets of terms which receive immediate interpretation in the Wheeler–Feynman theory. The first set is

$$\tfrac{1}{2}\sum_i \sum_j \int\int j_{(i)\mu}(x)\bar{D}(x-y)j_{(j)}{}^{\mu}(y)\,\mathrm{d}^4x\,\mathrm{d}^4y. \tag{25}$$

This will be recognized as the quantum action of the 'elementary' Wheeler–Feynman coupling between the charged currents. Comparison with the second term of equation (1) shows that we have merely replaced the classical currents with their corresponding operators j. There is a further important difference however. In the classical case the double summation ran over all $i \neq j$, but in (25) we must include a term $i = j$. That is, in the quantum theory a current may still act on itself. Although at first sight this seems to be against the spirit of the Wheeler–Feynman theory, it is clear that some structure of this sort must appear in the quantum theory in order that indistinguishable particles may interact without ambiguity. Moreover, if we are to interpret antiparticles as particles with reversed world lines (see figure 1), the known nonzero coupling between, for example, electrons and positrons, demands some sort of self-action.

Figure 1.

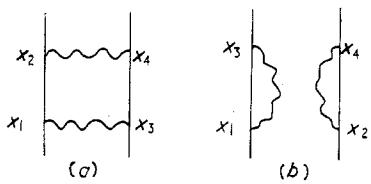

Figure 2.

Using the property (23), the second set of terms becomes

$$\tfrac{1}{4}\sum_i \sum_j \int\int j_{(i)\mu}(x)D^+(x-y)j_{(j)}{}^{\mu}(y)\,\mathrm{d}^4x\,\mathrm{d}^4y$$
$$+\tfrac{1}{4}\sum_i \sum_j \int\int j_{(i)\mu}(x)D^+(y-x)j_{(j)}{}^{\mu}(x)\,\mathrm{d}^4x\,\mathrm{d}^4y. \tag{26}$$

Because of the double summations both terms (26) are identical and we obtain, on adding

$$\tfrac{1}{2}\sum_i \sum_j \int\int j_{(i)\mu}(x)D^+(x-y)j_{(j)}{}^{\mu}(y)\,\mathrm{d}^4x\,\mathrm{d}^4y. \tag{27}$$

This set of terms may be interpreted as the response of the universe.

A simple way of seeing this is to adopt a model in which all the currents are considered as given classical functions. Then considering the contribution of (27) to the action of the ith species of current, we have

$$\int \mathrm{d}^4x J_{(i)}{}^{\mu}(x)\Big\{\sum_j \tfrac{1}{2}\int D^+(x-y)J_{(j)\mu}(y)\,\mathrm{d}^4y\Big\}. \tag{28}$$

Call the function in braces $A_\mu{}^{\mathrm{R}+}$. Then if the currents extend over the whole absorbing system $A_\mu{}^{\mathrm{R}+}$ will vanish everywhere by the same reasoning used in the classical theory above. In practice, we do not want the S matrix to include explicitly all interactions in the universe, but rather a small subset of interest in the laboratory. This is because if we localize our experimental apparatus we may consider all currents outside the laboratory to be distinguished, and label them by different letters in the summation in (27). As i, j now only run over a small subset, the contribution (27) is not zero and may be thought of as due to response fields $A^{\mathrm{R}+}_{(j)\mu}$ for each species j, in analogy to the classical fields (20).

A more rigorous demonstration may be given considering the currents as operators. Suppose we consider transitions between initial states α, which contain no free photons and unexcited absorber atoms, and final states β. The matrix S is given by expression (16) which is in general not unitary. This is because in arriving at (16) we ignored all processes involving real photons. However, if we now demand that the transition probability to final states with real photons is zero, (16) must be unitary. We therefore write the quantum absorber condition as

$$\sum_{\beta'} |\langle\beta'|S|\alpha\rangle|^2 = 0 \tag{29}$$

where β' are states containing real photons. Physically, condition (29) tells us that we have chosen our states to correspond to a system which completely absorbs all photons emitted.

Now (24) splits D_{F} into real ($\bar{D}$) and imaginary ($D^+ - D^-$) parts. This results in the decomposition of the integrand of (16) into Hermitean (25) and anti-Hermitean (27) parts. But if S is unitary, the integrand of (16) must be Hermitean, and so (27) will vanish when condition (29) is imposed on the states.

To summarize, conventional QED gives expression (16) whereas Wheeler–Feynman theory implies expression (25). But they lead to identical results when applied to a system in a light tight box, because their difference, expression (27), leaves the final results unchanged when appropriate states are taken for the S matrix.

Note that $A_\mu{}^{\mathrm{R}+}$ is the direct analogue of the classical response field (20) (when the currents are replaced by operators) except that in the quantum case we have singled out the positive frequencies of the D function. This is readily understood in the Wheeler–Feynman theory as an indication that the universe acts as an *absorber* of radiation, which means that only the positive frequencies which excite upward transitions of the particles in the cosmological medium, are completely absorbed. This

distinction cannot be made in the classical theory. This explains why we must use D_F in the quantum theory, but D_{ret} in the classical theory.

3. The Coulomb interaction

In all the above analysis, as well as in A it was implicit that we were dealing with transverse polarization directions only. This is essentially because in an electrically neutral universe there is no long range contribution to the Coulomb field, so we cannot have a response A_μ^R for $\mu = 4$. This need cause no difficulty though. There is no need to account for radiation of timelike (Coulomb) photons from a source system as there was in the case of the transverse polarization directions—an inevitable consequence of the existence of a response field. However, in a fully covariant treatment it is more elegant to treat all four polarization directions on an equal footing. Hence the Feynman propagator D_F is usually invoked even for Coulomb contributions, although it is known that we still obtain correct results working in the Coulomb gauge, with the transverse radiation field only being quantized. In the present theory Coulomb effects are propagated with $\bar{D}$, not D_F. It is easy to see that in practical calculations this makes no difference.

To see this, first note that when Coulomb forces are important we must integrate the particle wavefunctions over all space. In A, when we considered the interaction of bound particles, we were able to perform a two-centre decomposition which restricted the major contributions of these integrals to the region of emitter and absorber. Coulomb forces are then negligible. Only for free particles will they become important.

Consider lowest order Møller scattering. It is easy to show (eg Akhiezer and Berestetskii 1965, p. 515) that the conventional matrix element for scattering contains the factor

$$\int\int \frac{\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\exp(\mathrm{i}|E||\boldsymbol{r}_1-\boldsymbol{r}_2|)\,j_\mu^{\mathrm{AB}}(\boldsymbol{r}_1)j^{\mathrm{CD}\mu}(\boldsymbol{r}_2) \tag{30}$$

where the j are the transition currents between the free particle states A $\to$ B and C $\to$ D, while the exponential factor represents retardation over the interparticle distance $|\boldsymbol{r}_1-\boldsymbol{r}_2|$. E is the energy difference between A and B. If $E > 0$ then we may regard the retarded field of j_μ^{AB} to be acting on $j^{\mathrm{CD}\mu}$, or the advanced field of $j^{\mathrm{CD}\mu}$ acting the other way. If instead of D_F we had used $\bar{D}$, we would not have arrived at (30), but at

$$\tfrac{1}{2}\int\int \frac{\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\{\exp(\mathrm{i}|E||\boldsymbol{r}_1-\boldsymbol{r}_2|)+\exp(-\mathrm{i}|E||\boldsymbol{r}_1-\boldsymbol{r}_2|)\}j_\mu^{\mathrm{AB}}(\boldsymbol{r}_1)j^{\mathrm{CD}\mu}(\boldsymbol{r}_2). \tag{31}$$

It is straightforward to see that (30) and (31) give the same result when the integrations are carried out explicitly. To see this we assume the states A, B, and C, D are momentum eigenstates, in which case we are interested in the factors

$$\int\int \frac{\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\exp\{\mathrm{i}(\boldsymbol{p}_\mathrm{A}-\boldsymbol{p}_\mathrm{B})\cdot\boldsymbol{r}_1\}\exp\{-\mathrm{i}(\boldsymbol{p}_\mathrm{C}-\boldsymbol{p}_\mathrm{D})\cdot\boldsymbol{r}_2\}\exp(\pm\mathrm{i}|E||\boldsymbol{r}_1-\boldsymbol{r}_2|)$$

$$=\int\int \frac{\mathrm{d}\boldsymbol{r}_{12}\,\mathrm{d}\boldsymbol{r}_2}{r_{12}}\exp\{\mathrm{i}(\boldsymbol{p}_\mathrm{A}-\boldsymbol{p}_\mathrm{B})\cdot\boldsymbol{r}_{12}\}\exp\{-\mathrm{i}(\boldsymbol{p}_\mathrm{C}-\boldsymbol{p}_\mathrm{D}-\boldsymbol{p}_\mathrm{A}+\boldsymbol{p}_\mathrm{B})\cdot\boldsymbol{r}_2\}\exp(\pm\mathrm{i}|E|r_{12})$$

where

$$\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2.$$

The integration over $\boldsymbol{r}_2$ gives a momentum conserving δ function. The angular integration over $\hat{\boldsymbol{r}}_{12}$ gives a factor $\sin|\boldsymbol{p}_A-\boldsymbol{p}_B|r_{12}/|\boldsymbol{p}_A-\boldsymbol{p}_B|r_{12}$, so the factors of interest become

$$\int_0^\infty \frac{\mathrm{d}r_{12}}{2|\boldsymbol{p}_A-\boldsymbol{p}_B|}[\exp\{\mathrm{i}(|\boldsymbol{p}_A-\boldsymbol{p}_B| \pm |E|)r_{12}\} - \exp\{-\mathrm{i}(|\boldsymbol{p}_A-\boldsymbol{p}_B| \mp |E|)r_{12}\}].$$

These integrals may be evaluated using an appropriate convergence factor. If we use the + sign in front of $|E|$ (retarded case) we have

$$-\frac{1}{2|\boldsymbol{p}_A-\boldsymbol{p}_B|}\left(\frac{1}{|\boldsymbol{p}_A-\boldsymbol{p}_B|+|E|} - \frac{1}{-|\boldsymbol{p}_A-\boldsymbol{p}_B|+|E|}\right) = -\frac{1}{|\boldsymbol{p}_A-\boldsymbol{p}_B|^2-|E|^2}. \tag{32}$$

On the other hand, if we use $-|E|$ (advanced case) we have

$$-\frac{1}{2|\boldsymbol{p}_A-\boldsymbol{p}_B|}\left(\frac{1}{|\boldsymbol{p}_A-\boldsymbol{p}_B|-|E|} - \frac{1}{-|\boldsymbol{p}_A-\boldsymbol{p}_B|-|E|}\right) = -\frac{1}{|\boldsymbol{p}_A-\boldsymbol{p}_B|^2-|E|^2}. \tag{33}$$

Clearly, therefore, as regards the actual matrix elements, it is immaterial whether we use $\bar{D}$ or D_F.

Discussion

In 1948 Feynman demonstrated that in his path integral quantum mechanics the virtual photons could be eliminated from the description of electrodynamic processes. This resulted in an expression for the scattering amplitude involving an 'influence functional R' which only incorporates the direct current–current coupling through the propagator D_F. In this paper we have repeated Feynman's analysis using the interaction picture for the S matrix, which has the advantage of being a more familiar formalism. The result has been the same. The exponent of our equation (15) is the exact S matrix equivalent of Feynman's R. In a footnote following his equation (24), he points out that, from the classical theory 'one might have anticipated that R would have been simply R'' (R' is just R with D_F replaced by $\bar{D}$). He continues 'for a system enclosed in a light tight box, however, it appears likely that both R and R' lead to the same results'. In the present treatment changing R to R' merely corresponds to adding the quantity (27) to the exponent of (15). But in a light tight box we have seen that the additional terms due to D^+ leave the results unchanged and Feynman's conjecture is verified. Thus $\bar{D}$ and D_F give the same results in all nonradiative processes, provided we live in a universe which absorbs completely all positive frequency disturbances on the future light cone. Many cosmological models, for example, the Einstein–de Sitter model, do not satisfy this requirement.

Having established a general rule for all virtual photon processes Feynman also proceeded to demonstrate how real photon scattering amplitudes could be extracted from these formulae. This involves the question of separating out local processes from the rest of the universe. The elimination of the real photons will be dealt with in paper II.

References

AKHIEZER, A. I., and BERESTETSKII, V. B., 1965, *Quantum Electrodynamics* (New York: Interscience).
DAVIES, P. C. W., 1970, *Proc. Camb. Phil. Soc.*, **68,** 751–64.
FEYNMAN, R. P., 1948, *Rev. mod. Phys.*, **20,** 367–87.
—— 1950, *Phys. Rev.*, **80,** 440–57.
HOYLE, F., and NARKILAR, J. V., 1969, *Ann. Phys.*, **54,** 207–39.
—— 1971, *Ann. Phys.*, **62,** 44–97.
WHEELER, J. A., and FEYNMAN, R. P., 1945, *Rev. mod. Phys.*, **17,** 157–81.
—— 1949, *Rev. mod. Phys.*, **21,** 425–33.